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# ON CERTAIN HYPERABELIAN FUNCTIONS WHICH ARE EXPRESSIBLE BY THETA SERIES\*

BY

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The moduli of periodicity of an integral  $u_i$  of the first kind on the Riemann surface

$$y^{\nu} = (x - a)^{\alpha} (x - b)^{\beta} (x - c)^{\gamma} (x - d)^{\delta}$$

are linearly expressible in terms of two such  $A_i, B_i$  when  $\alpha + \beta + \gamma + \delta$  is a multiple of  $\nu$ . If  $u_i$  be replaced by the new integral  $u'_i = u_i/B_i$  the moduli of periodicity of  $u'_i$  will be of the form  $m + n\omega_i$  in which  $\omega_i = A_i/B_i$ . Accordingly the table of periods for the  $p$  integrals  $u'_i$  will be expressible linearly in terms of the  $p$  parameters  $\omega_i$ . The bilinear relations among the periods will enable us to reduce the linearly independent parameters  $\omega_i$  to a number  $p' < p$ .<sup>\*</sup> These remaining parameters will be connected by certain transcendental relations. Suppose, however, that after constructing the table of periods by aid of the Riemann surface, it is assumed that the  $p'$  linearly independent  $\omega_i$  are also absolutely independent of one another. The table of periods as thus generalized will no longer be related to the given Riemann surface. It can nevertheless serve as a table of theta moduli, since the necessary bilinear relations and inequality conditions are satisfied. The table being now written in its homogeneous form as expressed in terms of  $A_i$  and  $B_i$ , let these undergo the transformation

$$(1) \quad \begin{aligned} A'_i &= a_i A_i + b_i B_i, \\ B'_i &= c_i A_i + d_i B_i, \end{aligned}$$

and suppose that the result is equivalent to a linear transformation of the theta moduli the integer coefficients of which form the matrix

$$\begin{vmatrix} \alpha_{ik} & \beta_{ik} \\ \gamma_{ik} & \delta_{ik} \end{vmatrix}.$$

We thus obtain a group of transformations of the theta functions to which cor-

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<sup>†</sup> For more definite propositions concerning the relations among these  $\omega_i$ , see a paper presented to the American Mathematical Society, September 7, 1905, by Mr. RICHARD MORRIS, a brief résumé of which is given in the Bulletin for November, 1905.

responds a group of transformations on the  $\omega_i$  which is linear with respect to each variable. A uniform function of the  $\omega_i$  which is unaltered for the given group of transformations is called, after Picard, a *hyperabelian* function. Hyperabelian functions which belong to a group generated in the manner above described, can evidently be expressed in terms of theta functions with zero arguments.

To make the matter more clear, let us take as a particular case the Riemann surface

$$y^n = (x - a)(x - b)(x - c)^{n-1}(x - d)^{n-1}.$$

To form a convenient picture of the surface and its cross-cuts, let  $a, c, b, d$  be placed at the vertices of a rectangle in the order just written, the sides  $ac$  and  $bd$  of which are branch-lines of the surface. Take any point  $P$  (inside the rectangle, for convenience) and denote by  $\alpha_k$  a path starting at  $P$  in the  $k$ th sheet, winding once around  $a$  positively, and returning to  $P$  in the  $(k+1)$ th sheet. Let  $\beta_k$  be a similar path about  $b$ . Let  $\gamma_k$  be a path from  $P$  in the  $k$ th sheet, winding once around  $c$ , and returning in the  $(k-1)$ th sheet, and  $\delta_k$  a similar path about  $d$ . The equation

$$a_k = \sum_{i=1}^k \alpha_i + \sum_{i=k}^1 \beta_{i+1}$$

will be used to express symbolically that the path  $a_k$  is obtained by first describing the path  $\alpha_1$ , then the path  $\alpha_2$ , and so on in order. A canonical system of  $2p$  cross-cuts  $a_k, b_k$  may then be constructed from the  $a_k$  just defined and cuts  $b_k$  defined by the relation

$$b_k = \alpha_k + \delta_{k+1}.$$

If

$$u_k = \int \frac{(x - c)^{k-1}(x - d)^{k-1}}{y^k} dx \quad (k = 1, 2, \dots, n-1)$$

be taken as the  $p$  integrals of the first kind, the table of periods can readily be expressed in terms of the moduli of periodicity at  $a_1, b_1$ . If  $A_i, B_i$  denote the value of  $u_i$  when integrated along the paths  $a_1, b_1$  respectively, and if we write  $\rho = e^{2\pi i/5}$ , the result for  $n = 5$  is:

$$\begin{array}{l} \left. \begin{array}{l} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \end{array} \right| \begin{array}{l} -1, \quad -\rho^4, \quad -\rho^3, \quad -\rho^2 \\ -1, \quad -\rho^3, \quad -\rho, \quad -\rho^4 \\ -1, \quad -\rho^2, \quad -\rho^4, \quad -\rho \\ -1, \quad -\rho, \quad -\rho^2, \quad -\rho^3 \end{array} \left| \begin{array}{l} \omega_1, \quad (1 + \rho^4)\omega_1, \quad (-\rho - \rho^2)\omega_1, \quad -\rho\omega_1 \\ \omega_2, \quad (1 + \rho^3)\omega_2, \quad (-\rho^2 - \rho^4)\omega_2, \quad -\rho^2\omega_2 \\ \omega_3, \quad (1 + \rho^2)\omega_3, \quad (-\rho - \rho^3)\omega_3, \quad -\rho^3\omega_3 \\ \omega_4, \quad (1 + \rho)\omega_4, \quad (-\rho^3 - \rho^4)\omega_4, \quad -\rho^4\omega_4 \end{array} \right| \end{array}.$$

Instead of applying the processes indicated above to this table, we will deduce a simpler case by means of a transformation. Introduce as new integrals  $w_1 = \frac{1}{2}(u'_1 - u'_4)$ ,  $w_2 = \frac{1}{2}(u'_2 - u'_3)$ ,  $w_3 = \frac{1}{2}(u'_1 + u'_4)$ ,  $w_4 = \frac{1}{2}(u'_2 + u'_3)$  and make the transformation

1	0	1	1	0	
0	0	-1	-1		
1	2	1	1		
0	0	-1	1		
0				2	-1
				2	0
				0	1
				0	0
				0	0
				0	1
				0	0
				0	1

The above table then reduces to two hyperelliptic tables which, after multiplying the rows by  $B_1$  and  $B_2$  to make homogeneous and using the bilinear relations, are :

$$(I) \quad \begin{array}{l} W_1 B_1 \left| (\rho - \rho^4) B_1, (\rho^3 - \rho^2) B_2 \right| (1 + \rho^4) A_1, \rho^2 A_1 \\ W_2 B_2 \left| (\rho^2 - \rho^3) B_2, (\rho - \rho^4) B_2 \right| (1 + \rho^3) A_2, \rho^4 A_2 \end{array},$$

$$(II) \quad \begin{array}{l} W_3 B_1 \left| (\rho + \rho^4) B_1, (\rho^2 + \rho^3) B_1 \right| (1 - \rho^4) A_1, (1 + \rho - \rho^3 - \rho^4) A_1 \\ W_4 B_2 \left| (\rho^2 + \rho^3) B_2, (\rho + \rho^4) B_2 \right| (1 - \rho^3) A_2, (1 + \rho^2 - \rho - \rho^3) A_2 \end{array}.$$

In table (I) change  $A_i, B_i$  to  $A'_i, B'_i$  and substitute for these the expressions

(1) Again, denote the elements in (I) by the usual notation  $A_{ik} | B_{ik}$ , and let these be subjected to the transformation

$$A'_{ik} = \sum_{l=1}^2 (\alpha_{kl} A_{il} + \beta_{kl} B_{il}),$$

$$B'_{ik} = \sum_{l=1}^2 (\gamma_{kl} A_{il} + \delta_{kl} B_{il})$$

of determinant 1, in which  $\alpha_{ik}, \beta_{ik}, \dots$  are integers satisfying certain well known bilinear relations. Assume the two transformed tables thus obtained to be identical. On comparing like terms certain conditions are obtained from which the following results may be deduced.

For brevity write  $\alpha_{1i} = \alpha_i, \beta_{1i} = \beta_i, \gamma_{1i} = \gamma_i, \delta_{1i} = \delta_i$ . Then the theta transformation can be expressed in the form

$$\begin{array}{c|c} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_1 + \alpha_2 \\ \hline \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_1 + \gamma_2 \end{array} \quad \begin{array}{c|c} \beta_1 & \beta_2 \\ \beta_2 & \beta_1 + \beta_2 \\ \hline \delta_1 & \delta_2 \\ \delta_2 & \delta_1 + \delta_2 \end{array}$$

the coefficients being subject to the conditions

$$(2) \quad \begin{aligned} \alpha_1 \delta_1 + \alpha_2 \delta_2 - \beta_1 \gamma_1 - \beta_2 \gamma_2 &= 1, \\ \alpha_1 \delta_2 + \alpha_2 (\delta_1 + \delta_2) - \gamma_1 \beta_2 - \gamma_2 (\beta_1 + \beta_2) &= 0. \end{aligned}$$

The coefficients in the corresponding transformation on the  $\omega_i$

$$(3) \quad \omega'_i = \frac{a_i \omega_i + b_i}{c_i \omega_i + d_i} \quad (i = 1, 2)$$

take the form

$$\begin{aligned} a_1 &= \delta_1 - \lambda \delta_2, & a_2 &= \delta_1 - \lambda' \delta_2, \\ b_1 &= (\rho - 1)(\gamma_1 - \lambda \gamma_2), & b_2 &= (\rho^2 - 1)(\gamma_1 - \lambda' \gamma_2), \\ c_1 &= \frac{1}{\rho - 1}(\beta_1 - \lambda \beta_2), & c_2 &= \frac{1}{\rho^2 - 1}(\beta_1 - \lambda' \beta_2), \\ d_1 &= \alpha_1 - \lambda \alpha_2, & d_2 &= \alpha_1 - \lambda' \alpha_2, \end{aligned}$$

in which  $\lambda = \rho + \rho^4$  and  $\lambda' = \rho^2 + \rho^3$ . Also  $a_i d_i - b_i c_i = 1$  on account of (2). This transformation may be reduced to one with real coefficients by means of the substitution  $\omega_1 = i\rho^3 z_1, \omega_2 = i\rho z_2$ .

Reducing table (I) to the normal form in the usual way and applying the formulæ of KRAZER and PRYM for the transformation of the theta functions we find

$$\Delta_A = (c_1 \omega_1 + d_1)(c_2 \omega_2 + d_2)(\pi i)^2,$$

and hence obtain a relation of the form

$$(4) \quad \vartheta \left[ \begin{smallmatrix} g' \\ h' \end{smallmatrix} \right] (\omega'_1, \omega'_2) = C \sqrt{(c_1 \omega_1 + d_1)(c_2 \omega_2 + d_2)} \vartheta \left[ \begin{smallmatrix} g \\ h \end{smallmatrix} \right] (\omega_1, \omega_2),$$

in which  $C$  is a function of the coefficients of transformation only, and  $\vartheta \left[ \begin{smallmatrix} g \\ h \end{smallmatrix} \right] (\omega_1, \omega_2)$  denotes the theta function with zero arguments and moduli

$$\begin{aligned} a_{11} &= \frac{1}{5} \pi i [(\rho^4 - \rho)(1 + \rho^4) \omega_1 + (\rho^3 - \rho^2)(1 + \rho^3) \omega_2], \\ a_{12} &= \frac{1}{5} \pi i [(\rho - \rho^3) \omega_1 + (\rho^2 - \rho) \omega_2], \\ a_{22} &= \frac{1}{5} \pi i [(\rho^4 - 1) \omega_1 + (\rho^3 - 1) \omega_2]. \end{aligned}$$

By means of formula (4) we may construct from quotients of theta series, functions of  $\omega_1, \omega_2$  which are unaltered by the given group of transformations.

In a similar manner from table (II) we derive the theta transformations

$$\left| \begin{array}{cc|cc} \alpha_1 & \alpha_2 & \beta_1 & -\beta_2 \\ -\alpha_2 & \alpha_1 - 3\alpha_2 & -\beta_2 & -\beta_1 + 3\beta_2 \\ \hline \gamma_1 & \gamma_2 & \delta_1 & -\delta_2 \\ \gamma_2 & -\gamma_1 + 3\gamma_2 & \delta_2 & \delta_1 - 3\delta_2 \end{array} \right|,$$

the coefficients of which satisfy the conditions

$$\alpha_1 \delta_1 - \alpha_2 \delta_2 - \beta_1 \gamma_1 + \beta_2 \gamma_2 = 1,$$

$$\alpha_1 \delta_2 + \alpha_2 (\delta_1 - 3\delta_2) - \beta_2 \gamma_1 + \gamma_2 (-\beta_1 + 3\beta_2) = 0.$$

The coefficients in the transformation (3) are

$$a_1 = \delta_1 - \mu \delta_2,$$

$$a_2 = \delta_1 - \mu' \delta_2,$$

$$b_1 = \frac{1}{\rho - \rho^3} (\gamma_1 - \mu \gamma_2),$$

$$b_2 = \frac{1}{\rho^2 - \rho} (\gamma_1 - \mu' \gamma_2),$$

$$c_1 = (\rho - \rho^3)(\beta_1 - \mu \beta_2),$$

$$c_2 = (\rho^2 - \rho)(\beta_1 - \mu' \beta_2),$$

$$d_1 = \alpha_1 - \mu \alpha_2,$$

$$d_2 = \alpha_1 - \mu' \alpha_2,$$

in which  $\mu = \rho + \rho^4 + 2$  and  $\mu' = \rho^2 + \rho^3 + 2$ .

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